

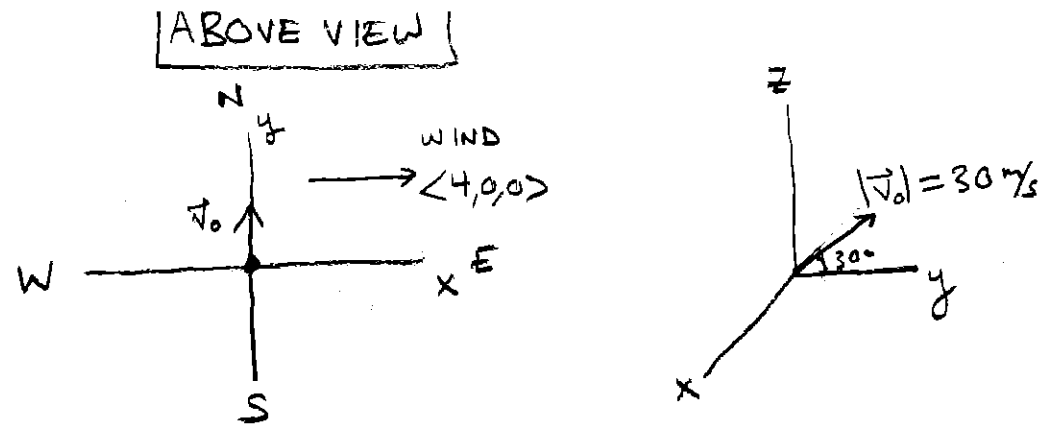
Close Tue: 10.3

Close Thu: 14.1, 14.3 (part 1)

Exam 1 will be returned Tuesday.

No office hours today (grading day).

Today: Finish 13.4 and start 10.3



Finishing 13.4 Acceleration/Velocity

Entry Task: A ball with mass $m = 0.8 \text{ kg}$ is thrown northward into the air with initial speed of 30 m/sec at an angle of 30 degrees with the ground. A west wind applies a steady force of 4 N on the ball (west to east).

If you are standing on level ground, where does the ball land?

FORCES?

$$\left. \begin{aligned} \vec{F}_g &= \langle 0, 0, -9.8 \text{ m} \rangle \\ \vec{F}_w &= \langle 4, 0, 0 \rangle \end{aligned} \right\} \vec{F} = \langle 4, 0, -9.8 \text{ m} \rangle$$

$$\vec{F} = m\vec{a} \Rightarrow \vec{a} = \frac{1}{m}\vec{F} = \left\langle \frac{4}{0.8}, 0, \frac{-9.8(0.8)}{0.8} \right\rangle$$

$$\vec{a}(t) = \langle 5, 0, -9.8 \rangle$$

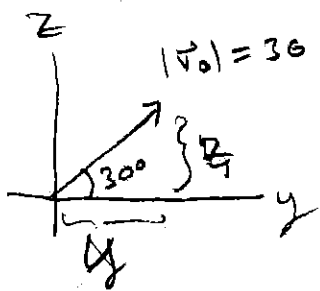
$$\vec{v}(t) = \langle 5t + c_1, c_2, -9.8t + c_3 \rangle$$

$$\vec{v}(t) = \langle 5t, 15\sqrt{3}, -9.8t + 15 \rangle$$

$$\vec{r}(t) = \left\langle \frac{5}{2}t^2 + d_1, 15\sqrt{3}t + d_2, -4.9t^2 + 15t + d_3 \right\rangle$$

$$\vec{r}(0) = 0 \Rightarrow d_1 = d_2 = d_3 = 0$$

$$\vec{r}(t) = \left\langle \frac{5}{2}t^2, 15\sqrt{3}t, -4.9t^2 + 15t \right\rangle$$



$$\left. \begin{aligned} \text{x-component} &= 0 \\ \text{y-component} &= 30 \cos(30^\circ) = 15\sqrt{3} \\ \text{z-component} &= 30 \sin(30^\circ) = 15 \end{aligned} \right\}$$

$$\vec{v}(0) = \langle 0, 15\sqrt{3}, 15 \rangle$$

HITS GROUND

WHEN

$$z = 0$$

$$\Leftrightarrow -4.9t^2 + 15t = 0$$

$$t(-4.9t + 15) = 0$$

$$t = 0 \quad \text{or} \quad -4.9t + 15 = 0$$

$$t = \frac{15}{4.9} \approx 3.06122 \text{ seconds}$$

$$x = \frac{5}{2} (t)^2 = 23.4277 \text{ m}$$

$$y = 15\sqrt{3} (t) = 79.5329 \text{ m}$$

$$z = 0$$



Example:

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

Find the tangential and normal components of acceleration.

$$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

$$|\vec{r}'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

$$\vec{r}''(t) = \langle -\cos(t), -\sin(t), 0 \rangle$$

$$\vec{r}' \cdot \vec{r}'' = \cos(t)\sin(t) - \cos(t)\sin(t) + 0 = 0$$

$$\Rightarrow a_T = \frac{\vec{r}' \cdot \vec{r}''}{|\vec{r}'|^2} = 0 \Rightarrow \text{CONSTANT SPEED}$$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin(t) & \cos(t) & 1 \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} = (0 - (-\sin(t)))\hat{i} - (0 - (-\cos(t)))\hat{j} + (\sin^2(t) - (-\cos^2(t)))\hat{k}$$
$$= \langle \sin(t), -\cos(t), 1 \rangle \quad \text{check } \checkmark$$

$$|\vec{r}' \times \vec{r}''| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

$$a_N = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^2} = \frac{\sqrt{2}}{\sqrt{2}} = 1 = \frac{v^2}{R} \Rightarrow R = \frac{1}{2}$$

RADIUS OF CURVATURE = 2

Deriving interpretations:

Note that: $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$

Let $v(t) = |\vec{v}(t)| = \text{speed}$.

$$1. \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\vec{v}(t)}{v(t)} \text{ implies } \vec{v} = v\vec{T}.$$

$$2. \kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{T}'|}{v(t)} \text{ implies } |\vec{T}'| = \kappa v.$$

$$3. \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{\vec{T}'}{\kappa v} \text{ implies } \vec{T}' = \kappa v \vec{N}.$$

Differentiating the first fact above gives

$$\vec{a} = \vec{v}' = v'\vec{T} + v\vec{T}', \text{ so}$$

$$\vec{a} = \vec{v}' = v'\vec{T} + \kappa v^2 \vec{N}.$$

Conclusion:

$$a_T = v' = \frac{d}{dt} |r'(t)| = \text{“deriv. of speed”}$$

$$a_N = \kappa v^2 = \text{curvature} \cdot (\text{speed})^2$$

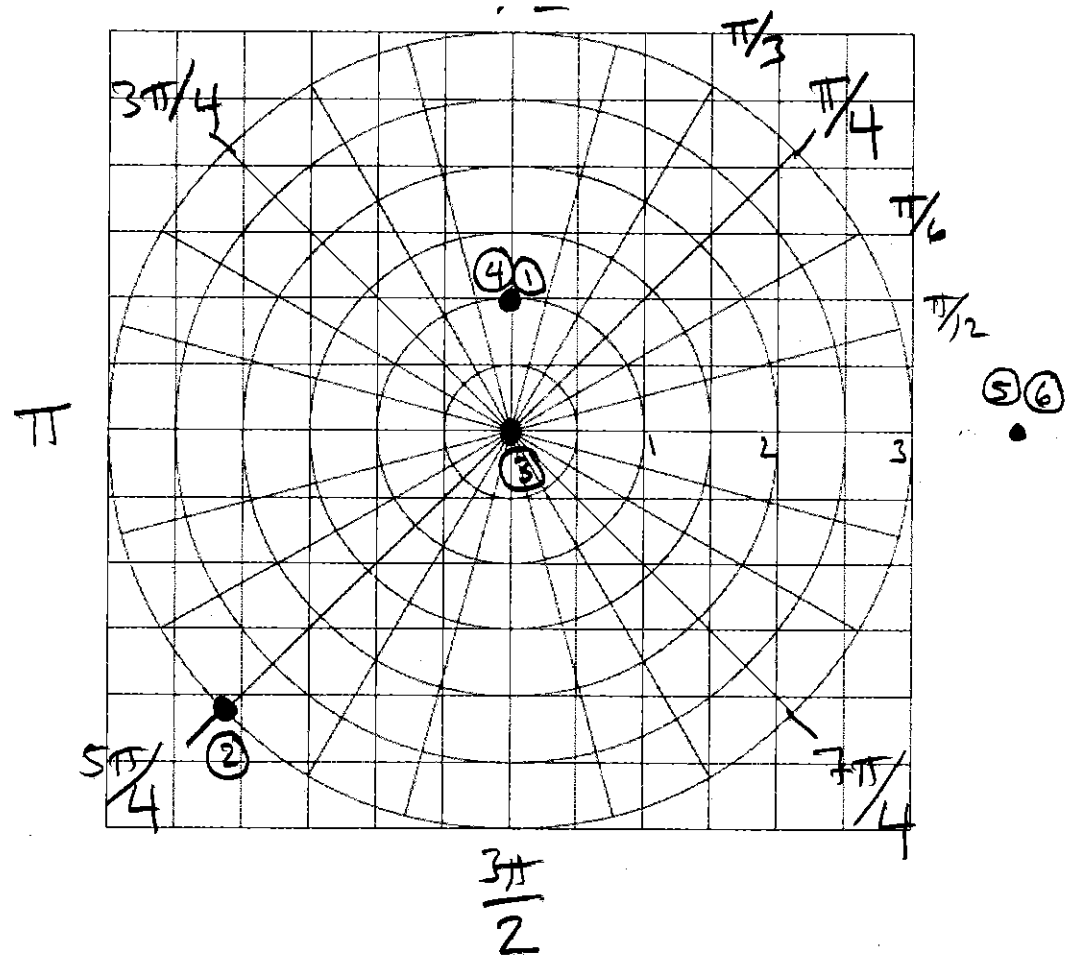
10.3 Polar Coordinates

Goal: A 2D coordinate system good for circular/arcing paths.

Cartesian	Polar
Given (x, y) 1. Stand at origin.	Given (r, θ) 1. Stand at origin facing the positive x -axis.
2. Move x -units on x -axis. pos. = right, neg. = left	2. Rotate by θ . pos. = ccw, neg. = clockwise
3. Move y -units parallel to y -axis. pos. = up neg. = down	3. Walk r -units in direction you are facing. pos. = forward neg. = backward

Example: Plot these polar points

- ① $(r, \theta) = (1, \pi/2)$
 - ② $(r, \theta) = (3, 5\pi/4)$
 - ③ $(r, \theta) = (0, \pi/3)$
 - ④ $(r, \theta) = (-1, 3\pi/2)$
 - ⑤ $(r, \theta) = (4, 0)$
 - ⑥ $(r, \theta) = (4, 100\pi)$
- SAME POINT
- SAME POINT



From trig we already know:

$$x = r \cos(\theta), \quad y = r \sin(\theta)$$

$$\tan(\theta) = \frac{y}{x}, \quad x^2 + y^2 = r^2$$

Exercise:

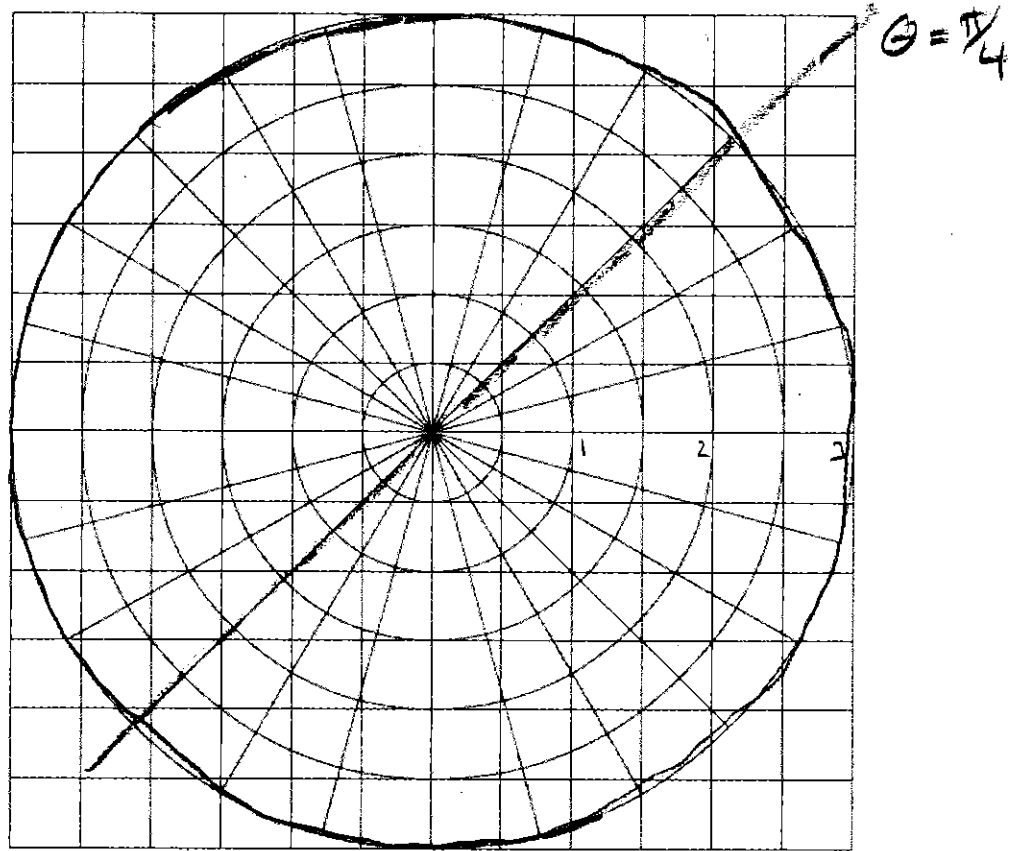
1. Describe all pts where $r = 3$.

$$\Rightarrow x = 3 \cos \theta \quad \text{AND} \quad x^2 + y^2 = 9$$
$$y = 3 \sin \theta \quad \text{CIRCLE!}$$

2. Describe all pts where $\theta = \pi/4$.

$$\theta = \pi/4, \quad r = \text{ANYTHING}$$
$$\tan(\pi/4) = \frac{y}{x}$$
$$1 \Rightarrow y = x$$

SAME



$$r = 3 \quad \theta = \text{ANYTHING}$$
$$x^2 + y^2 = 9$$

SAME

Plotting Polar Curves

Option 1: Try to convert to x and y .
Then hope you recognize the curve.

Option 2: Plot points!

Start with $0, \pi/2, \pi, 3\pi/2$ (intercepts).
For more detail do multiples of $\pi/6$
and $\pi/4$.

Example: Graph $r = \sin(\theta)$

$$r \sin \theta = y \Rightarrow \sin \theta = \frac{y}{r}$$

$$\Rightarrow r = \frac{y}{\sin \theta}$$

$$\Rightarrow r^2 = y$$

$$\Rightarrow x^2 + y^2 = y$$

$$\Rightarrow x^2 + y^2 - y = 0 \quad ??$$

COMPLETE SQUARE

$$x^2 + y^2 - y + \frac{1}{4} - \frac{1}{4} = 0$$

↳ HALF SQUARE

$$x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$$

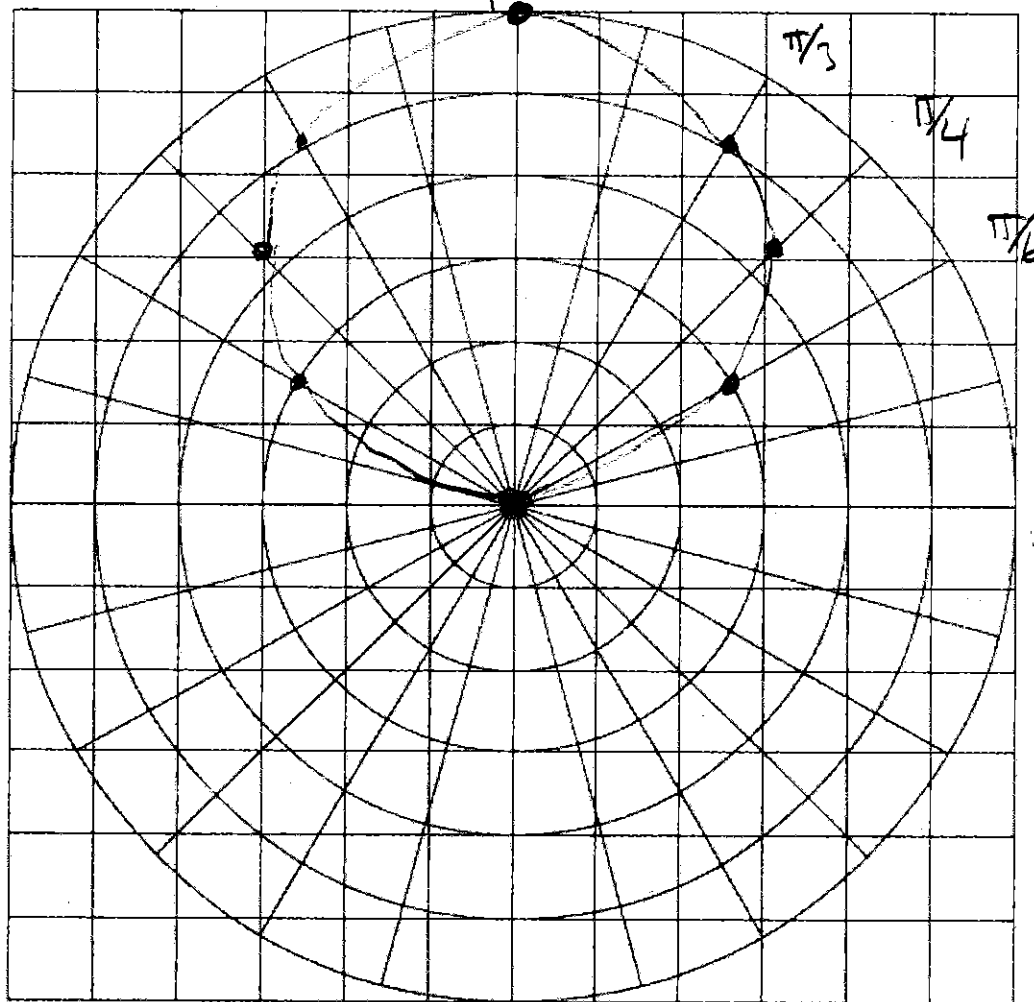
CENTER: $(0, \frac{1}{2})$

RADIUS: $\frac{1}{2}$

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	0	1	0	-1	0

θ	$\pi/6$	$\pi/4$	$\pi/3$	$2\pi/3$	$3\pi/4$	$5\pi/6$
r	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	

≈ 0.7 ≈ 0.86 ≈ 0.84

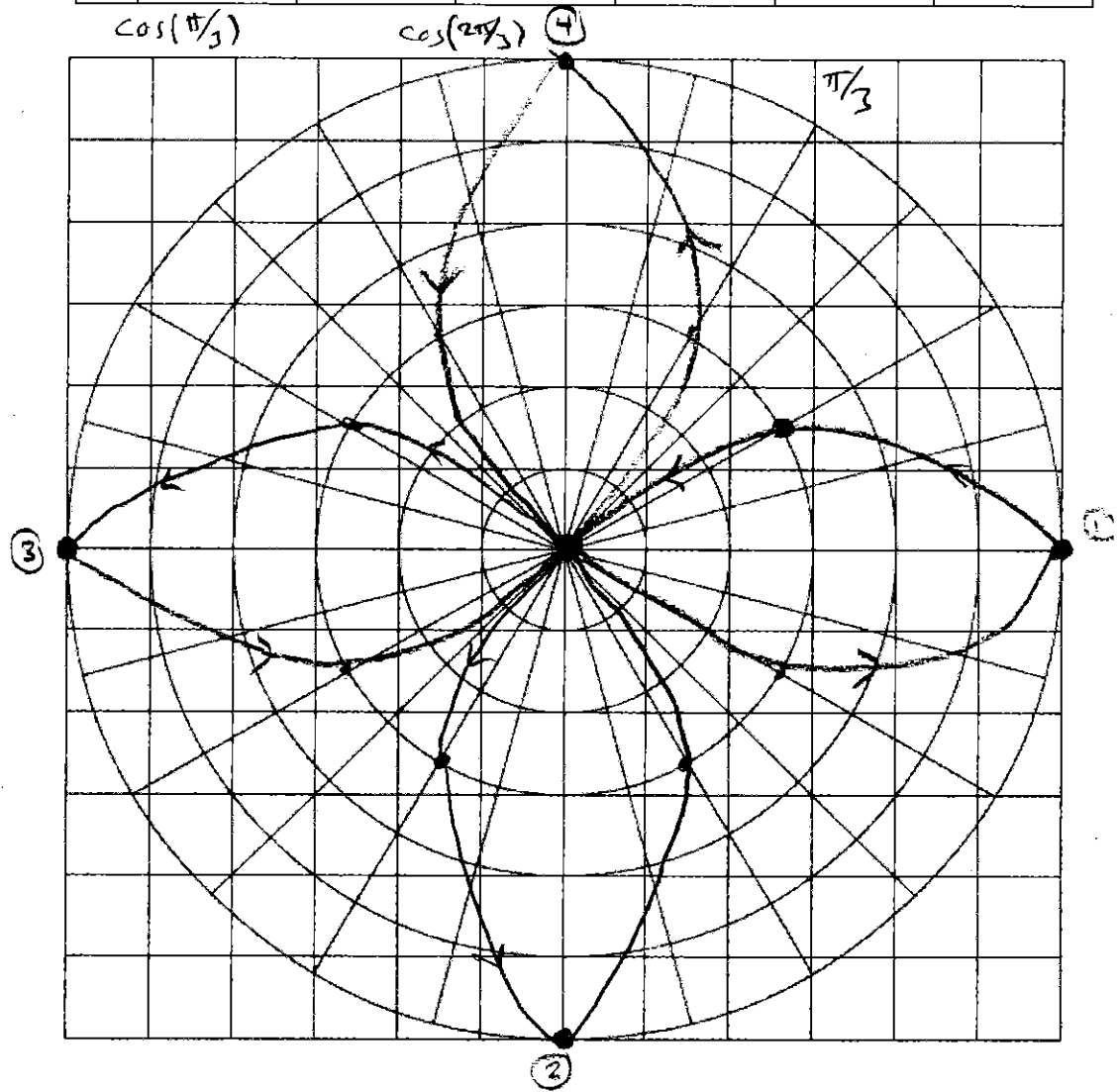


Example: Graph $r = \cos(2\theta)$

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	1	-1	1	-1	1

COULD USE HALF-ANGLE
 THEN TRY TO GET XY-EQUATION
 BUT IT WON'T BE NICE AND
 YOU WON'T RECOGNIZE
 THE XY-CURVE!
 SO JUST PLOT PTS

θ	$\pi/6$	$\pi/4$	$\pi/3$	$2\pi/3$	$3\pi/4$	$5\pi/6$
r	$1/2$	0	$-1/2$	$-1/2$	0	$1/2$



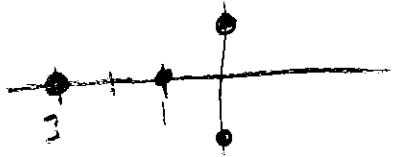
An old exam question:

The four polar equations below each match up with one of the six pictures. Identify which match.

1. $r = \sqrt{\theta}$ ← SPIRAL (r getting bigger)
2. $r = 1 - 2\cos(\theta)$
3. $r = 1 + \sin(2\theta)$
4. $r = 9\cos(\theta)$

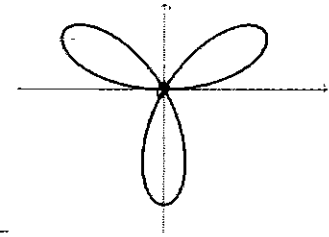
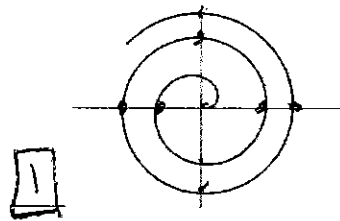
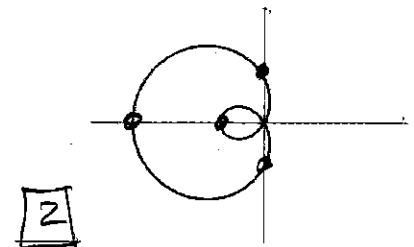
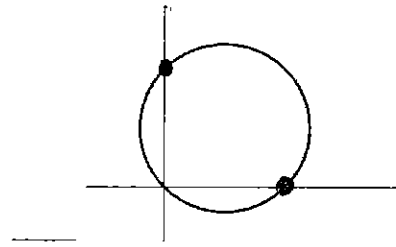
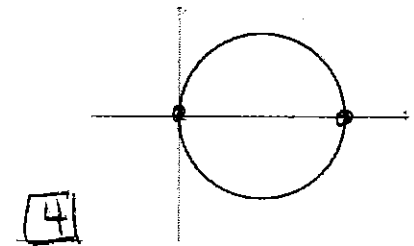
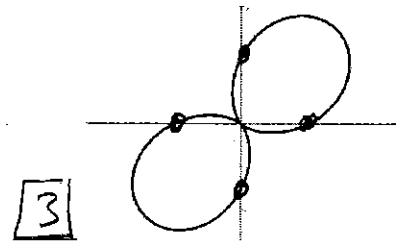
[2]

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
	-1	1	3	1



[3]

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
	1	1	1	1



[4]

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
	9	0	-9	0

